Closed sets

Def: Let X be a topological space. Then $A \subseteq X$ is closed if $X - A$ is open.

Ex: The set $[a,b]$ in $\mathbb{R}$ $(a \leq b)$ is closed since

$$\mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty) \text{ is open.}$$

Ex: If X has the cofinite topology, then the closed sets are all the finite sets and X itself.

Note: $\emptyset$ and X are always both open and closed! i.e. some subsets can be both.

Ex: If X is a set given the discrete topology, then all subsets are both open and closed.

Theorem: Let X be a topological space. Then

1.) $\emptyset$ and X are closed.

2.) Arbitrary intersections of closed sets are closed.

3.) Finite unions of closed sets are closed.

Pf: We've already shown 1.).

For 2.), let $\mathcal{C}$ be a collection of closed subsets of X. Define

$$V = \bigcap_{S \in \mathcal{C}} S.$$

Then $X - V = \bigcup_{S \in \mathcal{C}} (X - S)$, which is the union of open sets, and is thus open $\Rightarrow V$ is closed.

3.) follows a similar argument — try it yourself! $\square$

Def: Let X be a topological space, and $A \subseteq X$.

The <u>interior</u> of A, $\operatorname{Int} A$, is the union of the open sets contained in A.

The <u>closure</u> of A, $\bar{A}$, is the intersection of the closed sets containing A.

Remark: $\operatorname{Int} A$ is the union of open sets and is thus open. $\bar{A}$ is the intersection of closed sets and is thus closed.

A is open $\Leftrightarrow \operatorname{Int} A = A$. A is closed $\Leftrightarrow \bar{A} = A$.

Ex: Consider $\mathbb{R}$ with the standard topology.

Let $V = [0,1)$. Then $\operatorname{Int} V = (0,1)$:

We know $(0,1) \subseteq \operatorname{Int} V$ since $(0,1)$ is open.

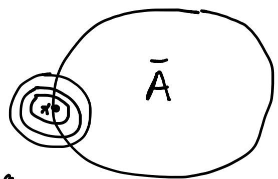
Thus $\operatorname{Int} V = (0,1)$ or $[0,1)$. $[0,1)$ is not open, so $\operatorname{Int} V = (0,1)$

By a similar argument, $\overline{V} = [0,1]$.

We need a better way to find the closure of sets:

Theorem: Let $A \subseteq X$. Then $x \in \bar{A} \iff \forall$ open sets U containing x, $U \cap A \neq \emptyset$.

Proof:



We prove this by proving the contrapositive in each direction:

$$x \notin \overline{A} \iff \exists\ U \subseteq X \text{ open s.t. } x \in U \text{ and } U \cap A = \emptyset.$$

First assume $x \notin \overline{A}$.

Then $x \in X - \overline{A}$, which is open and disjoint from A.

For the other direction, assume $U \subseteq X$ is open and $x \in U$ and $U \cap A = \emptyset$.

$X - U$ is closed and $A \subseteq X - U$, so $\overline{A} \subseteq X - U$. But $x \in U$, so $x \notin \overline{A}$. $\square$

Ex: Consider $\mathbb{R}$ w/ the standard topology.

- Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

  Any open set containing 0 will contain an open interval $(-\varepsilon, \varepsilon)$
  $\exists$ some $N \in \mathbb{Z}_+$ s.t. $\frac{1}{\varepsilon} < N \Rightarrow \frac{1}{N} < \varepsilon$.

Thus, $\frac{1}{N} \in (-\varepsilon, \varepsilon) \Rightarrow K \cap U \neq \emptyset$ for every open set U containing 0.

Thus $0 \in \overline{K}$. In fact, $\overline{K} = K \cup \{0\}$,

since $\mathbb{R} - (K \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup \left((\tfrac{1}{2}, 1) \cup (\tfrac{1}{3}, \tfrac{1}{2}) \cup \dots \right)$ is open, which means $K \cup \{0\}$ is closed.

Thus, $K \cup \{0\}$ is a closed set containing K, so it contains $\overline{K}$.

- Let $C \subseteq \mathbb{R}$ be finite. Let $V = \mathbb{R} - C$.

  Let $x \in C$. Then if $x \in U$, U open, then U contains infinitely many points, so $U \nsubseteq C$. Thus $U \cap V \neq \emptyset$, so $x \in \overline{V}$ $\forall$ $x \in C$.

  Thus $\overline{V} = \mathbb{R}$.

- If $C \subseteq \mathbb{R}$ is countable, $\overline{\mathbb{R} - C} = \mathbb{R}$. Use the fact that open intervals are uncountable. ((Can you prove this?)

Ex: Consider $\mathbb{R}$ w/ the K topology. Then $\mathbb{R} - K$ is open, so K is closed, so $\overline{K} = K$.

Ex: Consider $\mathbb{R}$ w/ the upper limit topology.

Then $(-\infty, 0]$ is open, so $\mathbb{R} - K$ is the union of open sets and is thus open. Thus K is closed, so $\overline{K} = K$.

Def: Let $A \subseteq X$. Then $x \in X$ is a <u>limit point</u> of A if every open set of X containing x intersects $A - \{x\}$. i.e. x is a limit point if $x \in \overline{A - \{x\}}$.

Theorem/Exercise: $\bar{A} = A \cup \{x \mid x \text{ is a limit point of } A\}$.

Corollary: A is closed $\iff$ A contains all its limit points.